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The Plasma Sheath in a Magnetized Plasma

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The Plasma Sheath in a Magnetized Plasma

I. INTRODUCTION

In this investigation we wish to determine the properties of the sheath formed about vehicles moving through the ionosphere. In particular, we wish to determine the particle density and potential distributions through the sheath. A theoretical derivation of the dependence of the sheath thickness on the medium parameters has not been given, so that we would also like to obtain some insight concerning this question from the analysis.

It might be thought that the answers to these questions could be obtained from a generalization of past work reported in the literature. It turns out, however, that the formulation used by those authors is not adequate to properly handle the present problem. For example, the theory of the plasma sheath which exists in a gas discharge was developed by Tonks and Langmuir in 1929 [1] for various geometries. We cannot, however, simply generalize their method to the case of a moving body in the presence of a magnetic field in order to handle the present problem. This is due to the fact that many assumptions were made in their work which, although are sufficiently good for applications to gas discharges, are not tenable for the present problem. By a similar argument, it is not possible to use the Boltzmann [2] or Fokker-Planck [3] transport equations in the present investigation for the reason that the derivation of these equations is not sufficiently general for our purposes. The derivation of the Boltzmann equation completely neglects the correlations between the particles in the plasma. This does not give a good description of a sheath because the large potential gradient makes such correlations important. Similarly, in the derivation of the Fokker-Planck

equation one assumes that the spatial gradient scales of the distribution functions are sufficiently large so that the spatial variation of the collision terms may be neglected. This assumption is not valid in the present investigation. In addition, the Fokker-Planck equation assumes that the collision process is Markovian. This assumption has been shown to be false for the case of an ionized gas by Tchen [4].

Watson [5] derived the Fokker-Planck equation including some space-dependence in the collision terms. His procedure has, however, been criticized by Ichikawa [6] on the basis that his approach is semi-phenomenological in that it depends upon a special model for the determination of the fluctuation force.

A method of formulating the sheath problem so as to include these space-dependent effects is inherent in Bogoliubov's method in statistical physics [7]. Tolmachev [8], Temko [9], and Tchen [4] used this method to derive the Fokker-Planck equation for a plasma. These authors neglected space-dependent effects, although Tchen investigated the non-Markovian behavior of the plasma, as we remarked above. Subsequently, Ichikawa [6] used a variant of Bogoliubov's method, introduced by Kadomtsev [10], to derive a Fokker-Planck equation which contained spatial dependent effects in the collision terms. Ichikawa emphasized the importance of the consideration of space-dependent terms in the equations describing the density fluctuations of a plasma (plasma oscillations), which had been neglected by previous authors.

II. DERIVATION OF THE EQUATIONS

We shall generalize the method used by Kadomtsev [10] and Ichikawa [6] to the case of a non-neutral two-component plasma in the presence of a static

magnetic field. Since the equations turn out to be so complicated, it will be better, for a first calculation, to neglect the vehicle motion.

First we consider a configuration-velocity space in which the coordinate axes are determined by the positions and velocities of all the particles at a given time t . If the system contains N_1 electrons and N_2 ions, then the configuration-velocity space has $6(N_1 + N_2)$ dimensions. We now define in this space the microscopic densities of the electrons and ions,

$$F_e = \sum_K \delta(\underline{x} - \underline{x}_K(\tau)) \delta(\underline{v} - \underline{v}_K(\tau)) \quad (1)$$

$$F_i = \sum_K \delta(\underline{x} - \underline{x}_K(\tau)) \delta(\underline{v} - \underline{v}_K(\tau)) \quad (2)$$

The indicated summations are over all electrons and ions respectively.

The quantities defined by (1) and (2) satisfy Liouville's theorem which states that the following relations hold:

$$\frac{d}{d\tau} F_e = \frac{d}{d\tau} F_i = 0 \quad (3)$$

Expansion of these equations gives:

$$\frac{2F_e}{2\tau} + \underline{v} \cdot \frac{2F_e}{2\underline{x}} + \frac{1}{m_e} \underline{G}_e \cdot \frac{2F_e}{2\underline{v}} = 0 \quad (4)$$

$$\frac{2F_i}{2\tau} + \underline{v} \cdot \frac{2F_i}{2\underline{x}} + \frac{1}{m_i} \underline{G}_i \cdot \frac{2F_i}{2\underline{v}} = 0 \quad (5)$$

where \underline{G}_e and \underline{G}_i are the microscopic forces per unit mass which act on the electrons and ions, respectively. We ignore the magnetic effects generated

by the motion of the charged particles, which is legitimate since the particles are non-relativistic, and write:

$$\underline{G}_e = -e \left[\underline{E}_m + \frac{1}{c} \underline{v} \times \underline{H} \right] \quad (6)$$

$$\underline{G}_i = e \left[\underline{E}_m + \frac{1}{c} \underline{v} \times \underline{H} \right] \quad (7)$$

where \underline{H} is the static external magnetic field and \underline{E}_m is the microscopic electric field acting on the particles. The microscopic field is given by:

$$\underline{E}_m(\underline{x}, \underline{z}) = -e \frac{2}{2\underline{x}} \iint \frac{[\underline{F}_i(\underline{x}', \underline{v}', \underline{z}) - \underline{F}_e(\underline{x}', \underline{v}', \underline{z})]}{|\underline{x} - \underline{x}'|} d\underline{x}' d\underline{v}' \quad (8)$$

Using (6), (7), and (8) in (4) and (5) we have:

$$\frac{2\underline{F}_e}{2\underline{z}} + \underline{v} \cdot \frac{2\underline{F}_e}{2\underline{x}} - \underline{v} \times \underline{\omega}_e \cdot \frac{2\underline{F}_e}{2\underline{v}} \quad (9)$$

$$- \frac{e^2}{m_e} \frac{2}{2\underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \underline{F}_e(\underline{z}) [\underline{F}_i(\underline{z}') - \underline{F}_e(\underline{z}')] d\underline{z}' = 0$$

$$\frac{2\underline{F}_i}{2\underline{z}} + \underline{v} \cdot \frac{2\underline{F}_i}{2\underline{x}} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{2\underline{F}_i}{2\underline{v}} \quad (10)$$

$$+ \frac{e^2}{m_i} \frac{2}{2\underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \underline{F}_i(\underline{z}) [\underline{F}_i(\underline{z}') - \underline{F}_e(\underline{z}')] d\underline{z}' = 0$$

where:

$$\begin{aligned}\omega_e &\equiv \frac{e}{m_e c} H \\ \alpha &\equiv \frac{m_e}{m_i} \\ \underline{r} &\equiv (\underline{x}, \underline{v}, \tau) \quad ; \quad \underline{r}' \equiv (\underline{x}', \underline{v}', \tau) \\ d\underline{r}' &\equiv d\underline{x}' d\underline{v}'\end{aligned}\tag{11}$$

It should be noted that, in performing the integrations in (9) and (10), the point $\underline{x} = \underline{x}'$ is to be omitted. This drops out the self-fields of the electrons and ions.

We now recall that the F 's are microscopic densities in the configuration-velocity space. Therefore, (9) and (10) are not soluble. Following the usual statistical mechanical treatment we shall average over the initial states of the system. We define:

$$\langle F_e \rangle = f_e \tag{12}$$

$$\langle F_i \rangle = f_i \tag{13}$$

where the brackets denote the statistical average. The procedure will now be to average equations (9) and (10) in the sense discussed above. To carry this out we will need expressions for the average of a product of two microscopic densities. A representative of this type of quantity is [10]:

$$\langle F_e(\underline{r}) F_e(\underline{r}') \rangle = \delta(\underline{r} - \underline{r}') f_e(\underline{r}) + f_{ee}(\underline{r}, \underline{r}') \tag{14}$$

where f_{ee} is called a binary distribution function. Actually, the first term of (14) will not contribute to the averages of (9) and (10) because the self-fields of the particles must be discarded. Averaging (9) and (10), we

find:

$$\frac{\partial f_e}{\partial \tau} + \underline{v} \cdot \frac{\partial f_e}{\partial \underline{x}} - \underline{v} \times \underline{\omega}_e \cdot \frac{\partial f_e}{\partial \underline{v}} \quad (15)$$

$$- \frac{e^2}{m_e} \frac{\partial}{\partial \underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [f_{ei}(\underline{r}, \underline{r}') - f_{ee}(\underline{r}, \underline{r}')] d\underline{r}' = 0$$

$$\frac{\partial f_i}{\partial \tau} + \underline{v} \cdot \frac{\partial f_i}{\partial \underline{x}} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{\partial f_i}{\partial \underline{v}} \quad (16)$$

$$+ \frac{e^2}{m_i} \frac{\partial}{\partial \underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [f_{ii}(\underline{r}, \underline{r}') - f_{ie}(\underline{r}, \underline{r}')] d\underline{r}' = 0$$

The inequality $e^2 n^{1/3} \ll kT$, where n is an average density of electrons or ions, and T is an appropriate electron or ion temperature, is satisfied in the present problem. Under these conditions we may write:

$$f_{ei}(\underline{r}, \underline{r}') = f_e(\underline{r}) f_i(\underline{r}') + \varphi_{ei}(\underline{r}, \underline{r}') \quad (17)$$

where the binary correlation function, φ , is a small quantity. This point has been discussed in more detail by Kadomtsev [10]. Using eqn. (17) and its variations, (15) and (16) now become:

$$\frac{\partial f_e}{\partial \tau} + \underline{v} \cdot \frac{\partial f_e}{\partial \underline{x}} - \underline{v} \times \underline{\omega}_e \cdot \frac{\partial f_e}{\partial \underline{v}} \quad (18)$$

$$- \frac{e^2}{m_e} \frac{\partial f_e}{\partial \underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [f_i(\underline{r}') - f_e(\underline{r}')] d\underline{r}'$$

$$- \frac{e^2}{m_e} \frac{\partial}{\partial \underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [\varphi_{ei}(\underline{r}, \underline{r}') - \varphi_{ee}(\underline{r}, \underline{r}')] d\underline{r}' = 0$$

$$\begin{aligned}
& \frac{2f_i}{2\tau} + \underline{v} \cdot \frac{2f_i}{2\underline{x}} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{2f_i}{2\underline{v}} \\
& + \frac{e^2}{m_i} \frac{2f_i}{2\underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [f_i(\underline{\Omega}') - f_e(\underline{\Omega}')] d\underline{\Omega}' \\
& + \frac{e^2}{m_i} \frac{2}{2\underline{v}} \cdot \iint \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} [\varphi_{ii}(\underline{\Omega}, \underline{\Omega}') - \varphi_{ie}(\underline{\Omega}, \underline{\Omega}')] d\underline{\Omega}' = 0
\end{aligned} \tag{19}$$

The next to last terms in (18) and (19) do not appear in the corresponding equations of Kadomtsev. This is because he assumes a neutral plasma, although this is not explicitly stated. We cannot drop these terms, however, because the sheath is not a neutral region. On the other hand, although the φ 's are small, we cannot completely neglect them because they are important for large and small values of $|\underline{x} - \underline{x}'|$. This is related to the problem of the divergence of Coulomb integrals. Wu and Rosenberg, however, have argued [11] that, in fact, there is no "divergence difficulty" if the matter is handled in the proper fashion.

Thus, the next step is to derive equations for the binary correlation functions φ . The technique is to multiply (9) or (10) by an F , then multiply the unused equation (either (9) or (10)) by an F (either the same or different from the first one), then add the two resulting equations, and then perform the statistical average on this equation. It is now found that the first term of (14) contributes to the averaged equation. In addition, we must have a method of evaluating the statistical average of a product of three microscopic densities. An example of such an average is [6]:

$$\begin{aligned}
\langle F_e(\underline{r}) F_e(\underline{r}') F_e(\underline{r}'') \rangle &= \delta(\underline{r} - \underline{r}') f_{ee}(\underline{r}'', \underline{r}') \\
&+ \delta(\underline{r}' - \underline{r}'') f_{ee}(\underline{r}'', \underline{r}) + \delta(\underline{r} - \underline{r}'') f_{ee}(\underline{r}, \underline{r}') \\
&+ f_{eee}(\underline{r}, \underline{r}', \underline{r}'')
\end{aligned} \tag{20}$$

Following the same argument that led to (17) we may write the ternary distribution functions in the form [6]:

$$\begin{aligned}
f_{eee}(\underline{r}, \underline{r}', \underline{r}'') &= f_e(\underline{r}) f_e(\underline{r}') f_e(\underline{r}'') \\
&+ f_e(\underline{r}) \varphi_{ee}(\underline{r}', \underline{r}'') + f_e(\underline{r}') \varphi_{ee}(\underline{r}'', \underline{r}) \\
&+ f_e(\underline{r}'') \varphi_{ee}(\underline{r}, \underline{r}') + \psi(\underline{r}, \underline{r}', \underline{r}'')
\end{aligned} \tag{21}$$

Now, Bogoliubov [7] and Tchen [4] have shown that the ternary correlation function, ψ , may be neglected when the inequality given under eqn. (16) holds. There is one more point to talk about before writing down the equations for the binary correlation functions. This is that we may set $\varphi_{ei} = \varphi_{ie}$ in (18) and (19). The statement is obvious on physical grounds. Mathematically, it follows from the fact that the exact, microscopic densities must be symmetric functions of their arguments. Compare the discussion by Bogoliubov [7].

Now, following the procedure described above making use of (14), (17), (20), (21) and their variations we find the following three equations for the binary correlation functions:

$$\mathcal{O}_1 \varphi_{ee}(\underline{r}, \underline{r}'') + \frac{1}{m_e} \iint \underline{c}(\underline{r}, \underline{r}') \cdot \frac{\partial}{\partial \underline{r}} \left\{ \delta(\underline{r} - \underline{r}'') \left[\varphi_{ee}(\underline{r}, \underline{r}'') - \varphi_{ei}(\underline{r}, \underline{r}'') \right] \right\} d\underline{r}' \tag{22}$$

$$\begin{aligned}
& + \frac{1}{m_e} \frac{\partial f_e(\underline{r})}{\partial \underline{v}} \cdot \iint \underline{c}(\underline{x}, \underline{x}') \left[\rho_{ee}(\underline{r}', \underline{r}'') - \rho_{ie}(\underline{r}', \underline{r}'') \right] d\underline{r}' \\
& - \frac{1}{m_e} \underline{M}(\underline{x}) \cdot \frac{\partial \rho_{ee}(\underline{r}, \underline{r}'')}{\partial \underline{v}} + \frac{1}{m_e} f_e(\underline{r}'') \iint \underline{c}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left[\rho_{ee}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] d\underline{r}' \\
& + \frac{1}{m_e} \underline{c}(\underline{x}'', \underline{x}) \cdot \frac{\partial}{\partial \underline{v}} \left[\rho_{ee}(\underline{r}'', \underline{r}) - \rho_{ie}(\underline{r}'', \underline{r}) \right] \\
& + \frac{1}{m_e} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left\{ \delta(\underline{r} - \underline{r}'') \left[\rho_{ee}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] \right\} d\underline{r}' \\
& + \frac{f_e(\underline{r})}{m_e} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left[\rho_{ee}(\underline{r}', \underline{r}'') - \rho_{ie}(\underline{r}', \underline{r}'') \right] d\underline{r}' \\
& + \frac{1}{m_e} \frac{\partial f_e(\underline{r}'')}{\partial \underline{v}''} \cdot \iint \underline{c}(\underline{x}'', \underline{x}') \left[\rho_{ee}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] d\underline{r}' \\
& - \frac{1}{m_e} \underline{M}(\underline{x}'') \cdot \frac{\partial \rho_{ee}(\underline{r}'', \underline{r})}{\partial \underline{v}''} = -A_1(\underline{r}, \underline{r}'')
\end{aligned} \tag{22}$$

$$\begin{aligned}
& \partial_2 \rho_{ii}(\underline{r}, \underline{r}'') + \frac{1}{m_i} \iint \underline{c}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left\{ \delta(\underline{r} - \underline{r}'') \left[\rho_{ii}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] \right\} d\underline{r}' \\
& + \frac{1}{m_i} \frac{\partial f_i(\underline{r})}{\partial \underline{v}} \cdot \iint \underline{c}(\underline{x}, \underline{x}') \left[\rho_{ii}(\underline{r}', \underline{r}'') - \rho_{ie}(\underline{r}', \underline{r}'') \right] d\underline{r}' \\
& + \frac{f_i(\underline{r}'')}{m_i} \iint \underline{c}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left[\rho_{ii}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] d\underline{r}' \\
& + \frac{1}{m_i} \underline{M}(\underline{x}) \cdot \frac{\partial \rho_{ii}(\underline{r}'', \underline{r})}{\partial \underline{v}} + \frac{1}{m_i} \underline{c}(\underline{x}'', \underline{x}) \cdot \frac{\partial}{\partial \underline{v}''} \left[\rho_{ii}(\underline{r}'', \underline{r}) - \rho_{ie}(\underline{r}'', \underline{r}) \right] \\
& + \frac{1}{m_i} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left\{ \delta(\underline{r} - \underline{r}'') \left[\rho_{ii}(\underline{r}, \underline{r}') - \rho_{ie}(\underline{r}, \underline{r}') \right] \right\} d\underline{r}'
\end{aligned} \tag{23}$$

$$\begin{aligned}
& + \frac{f_i(\underline{r})}{m_i} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} [\varphi_{ii}(\underline{r}', \underline{r}'') - \varphi_{ie}(\underline{r}', \underline{r}'')] d\underline{r}' \\
& + \frac{1}{m_i} \frac{\partial f_i(\underline{r}'')}{\partial \underline{v}''} \cdot \iint \underline{c}(\underline{x}'', \underline{x}') [\varphi_{ii}(\underline{r}, \underline{r}') - \varphi_{ie}(\underline{r}, \underline{r}')] d\underline{r}' \\
& + \frac{1}{m_i} M(\underline{x}'') \cdot \frac{\partial \varphi_{ii}(\underline{r}'', \underline{r})}{\partial \underline{v}''} = -A_2(\underline{r}, \underline{r}'')
\end{aligned}$$

$$\begin{aligned}
& \theta_3 \varphi_{ei}(\underline{r}, \underline{r}'') + \frac{1}{m_e} \iint \underline{c}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left\{ \delta(\underline{r} - \underline{r}'') [\varphi_{ee}(\underline{r}, \underline{r}') - \varphi_{ei}(\underline{r}, \underline{r}')] \right\} d\underline{r}' \\
& - \frac{1}{m_e} \frac{\partial f_e(\underline{r})}{\partial \underline{v}} \cdot \iint \underline{c}(\underline{x}, \underline{x}') [\varphi_{ii}(\underline{r}', \underline{r}') - \varphi_{ei}(\underline{r}', \underline{r}'')] d\underline{r}' \\
& - \frac{1}{m_e} M(\underline{x}) \cdot \frac{\partial \varphi_{ie}(\underline{r}'', \underline{r})}{\partial \underline{v}} + \frac{f_i(\underline{r}'')}{m_e} \iint \underline{c}(\underline{x}, \underline{x}') [\varphi_{ee}(\underline{r}, \underline{r}') - \varphi_{ei}(\underline{r}, \underline{r}')] d\underline{r}' \\
& - \frac{1}{m_e} \underline{c}(\underline{x}, \underline{x}'') \cdot \frac{\partial}{\partial \underline{v}} [\varphi_{ii}(\underline{r}'', \underline{r}) - \varphi_{ie}(\underline{r}'', \underline{r})] \quad (24) \\
& - \frac{1}{m_i} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left\{ \delta(\underline{r} - \underline{r}'') [\varphi_{ee}(\underline{r}, \underline{r}') - \varphi_{ei}(\underline{r}, \underline{r}')] \right\} d\underline{r}' \\
& + \frac{f_e(\underline{r})}{m_i} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} [\varphi_{ii}(\underline{r}', \underline{r}'') - \varphi_{ei}(\underline{r}', \underline{r}'')] d\underline{r}' \\
& + \frac{1}{m_i} M(\underline{x}'') \cdot \frac{\partial \varphi_{ie}(\underline{r}'', \underline{r})}{\partial \underline{v}''} - \frac{1}{m_i} \frac{\partial f_i(\underline{r}'')}{\partial \underline{v}''} \cdot \iint \underline{c}(\underline{x}'', \underline{x}') [\varphi_{ee}(\underline{r}, \underline{r}') - \varphi_{ei}(\underline{r}, \underline{r}')] d\underline{r}' \\
& + \frac{1}{m_i} \underline{c}(\underline{x}'', \underline{x}) \cdot \frac{\partial}{\partial \underline{v}''} [\varphi_{ii}(\underline{r}'', \underline{r}) - \varphi_{ie}(\underline{r}'', \underline{r})] = -A_3(\underline{r}, \underline{r}'')
\end{aligned}$$

Although the three equations (22)-(24) form a complete set when combined with (18) and (19), we now write down the equation for φ_{ie} (even though $\varphi_{ie} = \varphi_{ei}$). We do this because the properties of the particles - i.e., distribution functions, masses - appear in a different way in the two equations. Even though the two functions are the same, the equations determining them are not. It is difficult at this state of the analysis, particularly since the equations are so complicated, to know the proper combination of equations to use so as to minimize the labor involved in solving them. It may prove convenient later on to use a combination of eqns. (24) and (25) instead of either one separately. We have:

$$\begin{aligned}
 & \mathcal{O}_4 \varphi_{ie}(\underline{z}, \underline{z}'') + \frac{1}{m_i} \iint \underline{c}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left\{ \delta(\underline{z} - \underline{z}') \left[\varphi_{ie}(\underline{z}, \underline{z}') - \varphi_{ie}(\underline{z}', \underline{z}') \right] \right\} d\underline{z}' \\
 & - \frac{1}{m_i} \frac{\partial f_i(\underline{z})}{\partial \underline{v}} \cdot \iint \underline{c}(\underline{x}, \underline{x}') \left[\varphi_{ee}(\underline{z}', \underline{z}'') - \varphi_{ie}(\underline{z}', \underline{z}'') \right] d\underline{z}' \\
 & + \frac{1}{m_i} M(\underline{x}) \cdot \frac{\partial \varphi_{ie}(\underline{z}, \underline{z}'')}{\partial \underline{v}} + \frac{f_e(\underline{z}'')}{m_i} \iint \underline{c}(\underline{z}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} \left[\varphi_{ie}(\underline{z}, \underline{z}') - \varphi_{ie}(\underline{z}', \underline{z}') \right] d\underline{z}' \\
 & - \frac{1}{m_e} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left\{ \delta(\underline{z} - \underline{z}'') \left[\varphi_{ie}(\underline{z}, \underline{z}') - \varphi_{ie}(\underline{z}', \underline{z}'') \right] \right\} d\underline{z}' \\
 & + \frac{f_i(\underline{z})}{m_e} \iint \underline{c}(\underline{x}'', \underline{x}') \cdot \frac{\partial}{\partial \underline{v}''} \left[\varphi_{ee}(\underline{z}', \underline{z}'') - \varphi_{ie}(\underline{z}', \underline{z}'') \right] d\underline{z}'
 \end{aligned}
 \tag{25}$$

$$\begin{aligned}
& - \frac{1}{m_e} \frac{\partial f_e(\underline{r}'')}{\partial \underline{v}''} \cdot \left(\int \underline{c}(\underline{x}'', \underline{x}') [\varphi_{ii}(\underline{r}, \underline{r}') - \varphi_{ie}(\underline{r}, \underline{r}')] d\underline{r}' \right. \\
& + \frac{1}{m_e} \underline{c}(\underline{x}'', \underline{x}) \cdot \frac{\partial}{\partial \underline{v}''} [\varphi_{ee}(\underline{r}'', \underline{r}) - \varphi_{ei}(\underline{r}'', \underline{r})] \\
& \left. - \frac{1}{m_e} \underline{M}(\underline{x}'') \cdot \frac{\partial \varphi_{ei}(\underline{r}'', \underline{r})}{\partial \underline{v}''} \right) = -A_4(\underline{r}, \underline{r}'')
\end{aligned} \tag{25}$$

The undefined quantities which appear in eqns. (22)-(25) are listed below.

$$\begin{aligned}
A_1(\underline{r}, \underline{r}'') & \equiv \varphi_i [\delta(\underline{r} - \underline{r}'') f_e(\underline{r}) + f_e(\underline{r}) f_e(\underline{r}'')] \\
& - \frac{1}{m_e} \underline{M}(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} \{ \delta(\underline{r} - \underline{r}'') f_e(\underline{r}) \} \\
& - \frac{1}{m_e} f_e(\underline{r}'') \frac{\partial f_e(\underline{r})}{\partial \underline{v}} \cdot \underline{M}(\underline{x}) - \frac{1}{m_e} [f_i(\underline{r}) - f_e(\underline{r})] \frac{\partial f_e(\underline{r}'')}{\partial \underline{v}''} \cdot \underline{c}(\underline{x}'', \underline{x}) \\
& - \frac{1}{m_e} \underline{M}(\underline{x}'') \cdot \frac{\partial}{\partial \underline{v}''} [\delta(\underline{r} - \underline{r}'') f_e(\underline{r})] \\
& - \frac{1}{m_e} f_e(\underline{r}) \frac{\partial f_e(\underline{r}'')}{\partial \underline{v}''} \cdot \underline{M}(\underline{x}'')
\end{aligned} \tag{26}$$

$$\begin{aligned}
A_2(\underline{x}, \underline{x}'') &\equiv \frac{1}{m_i} \frac{\partial f_i(\underline{x}'')}{\partial \underline{v}''} \cdot \underline{\zeta}(\underline{x}'', \underline{x}) [f_i(\underline{x}) - f_e(\underline{x})] \\
&+ \mathcal{O}_2 [\delta(\underline{x} - \underline{x}'') f_i(\underline{x}) + f_i(\underline{x}) f_i(\underline{x}'')] \\
&+ \frac{1}{m_i} M(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} [\delta(\underline{x} - \underline{x}'') f_i(\underline{x})] + \frac{1}{m_i} f_i(\underline{x}'') \frac{\partial f_i(\underline{x})}{\partial \underline{v}} \cdot M(\underline{x})_{(27)} \\
&+ \frac{1}{m_i} M(\underline{x}'') \cdot \frac{\partial}{\partial \underline{v}''} [\delta(\underline{x} - \underline{x}'') f_i(\underline{x})] + \frac{1}{m_i} f_i(\underline{x}) \frac{\partial f_i(\underline{x}'')}{\partial \underline{v}''} \cdot M(\underline{x}'')
\end{aligned}$$

$$\begin{aligned}
A_3(\underline{x}, \underline{x}'') &\equiv \mathcal{O}_3 [\delta(\underline{x} - \underline{x}'') f_e(\underline{x}) + f_e(\underline{x}) f_e(\underline{x}'')] \\
&- \frac{1}{m_e} f_i(\underline{x}'') \underline{\zeta}(\underline{x}, \underline{x}'') \cdot \frac{\partial}{\partial \underline{v}} [f_i(\underline{x}) - f_e(\underline{x})] \\
&- \frac{1}{m_e} M(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} [\delta(\underline{x} - \underline{x}'') f_e(\underline{x})] \quad (28) \\
&- \frac{1}{m_e} f_i(\underline{x}'') \frac{\partial f_e(\underline{x})}{\partial \underline{v}} \cdot M(\underline{x}) + \frac{1}{m_i} f_e(\underline{x}) \frac{\partial f_i(\underline{x}'')}{\partial \underline{v}''} \cdot M(\underline{x}'') \\
&+ \frac{1}{m_i} M(\underline{x}'') \cdot \frac{\partial}{\partial \underline{v}''} [\delta(\underline{x} - \underline{x}'') f_e(\underline{x})] \\
&+ \frac{1}{m_i} [f_i(\underline{x}) - f_e(\underline{x})] \frac{\partial f_i(\underline{x}'')}{\partial \underline{v}''} \cdot \underline{\zeta}(\underline{x}'', \underline{x})
\end{aligned}$$

$$\begin{aligned}
A_4(\underline{x}, \underline{x}'') &\equiv \partial_4 \left[\delta(\underline{x} - \underline{x}'') f_i(\underline{x}) + f_i(\underline{x}) f_e(\underline{x}'') \right] \\
&+ \frac{1}{m_i} f_e(\underline{x}'') \frac{\partial f_i(\underline{x})}{\partial \underline{v}} \cdot \underline{M}(\underline{x}) - \frac{1}{m_e} \underline{M}(\underline{x}'') \cdot \frac{\partial}{\partial \underline{v}''} \left[\delta(\underline{x} - \underline{x}'') f_i(\underline{x}) \right] \\
&+ \frac{1}{m_i} \underline{M}(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} \left[\delta(\underline{x} - \underline{x}'') f_i(\underline{x}) \right] \quad (29) \\
&- \frac{1}{m_e} f_i(\underline{x}) \frac{\partial f_e(\underline{x}'')}{\partial \underline{v}''} \cdot \underline{M}(\underline{x}'') - \frac{1}{m_e} [f_i(\underline{x}) - f_e(\underline{x})] \frac{\partial f_e(\underline{x}'')}{\partial \underline{v}''} \cdot \underline{c}(\underline{x}'', \underline{x})
\end{aligned}$$

where:

$$\underline{c}(\underline{x}, \underline{x}') \equiv e^2 \frac{(\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3} \quad (30)$$

$$\underline{M}(\underline{x}) \equiv \iint \underline{c}(\underline{x}, \underline{x}') [f_i(\underline{x}') - f_e(\underline{x}')] d\underline{x}' \quad (31)$$

$$\partial_1 \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{v}'' \cdot \frac{\partial}{\partial \underline{x}''} - \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} - \underline{v}'' \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}''} \quad (32)$$

$$\partial_2 \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{v}'' \cdot \frac{\partial}{\partial \underline{x}''} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} + \alpha \underline{v}'' \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}''} \quad (33)$$

$$\partial_3 \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{v}'' \cdot \frac{\partial}{\partial \underline{x}''} - \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} + \alpha \underline{v}'' \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}''} \quad (34)$$

$$\mathcal{O}_4 \equiv \frac{\partial}{\partial \tau} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \underline{v}'' \cdot \frac{\partial}{\partial \underline{x}''} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} - \underline{v}'' \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}''} \quad (35)$$

For completeness we now rewrite (18) and (19) in a more compact form:

$$N_1(\underline{\Omega}) + \frac{1}{m_e} \iint \underline{C}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} [\varphi_{ee}(\underline{\Omega}, \underline{\Omega}') - \varphi_{ei}(\underline{\Omega}, \underline{\Omega}')] d\underline{\Omega}' = 0 \quad (36)$$

$$N_2(\underline{\Omega}) + \frac{1}{m_i} \iint \underline{C}(\underline{x}, \underline{x}') \cdot \frac{\partial}{\partial \underline{v}} [\varphi_{ii}(\underline{\Omega}, \underline{\Omega}') - \varphi_{ie}(\underline{\Omega}, \underline{\Omega}')] d\underline{\Omega}' = 0 \quad (37)$$

where:

$$N_1(\underline{\Omega}) \equiv \left(\frac{\partial}{\partial \tau} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} - \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} - \frac{1}{m_e} \underline{M}(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} \right) f_e(\underline{\Omega}) \quad (38)$$

$$N_2(\underline{\Omega}) \equiv \left(\frac{\partial}{\partial \tau} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}} + \alpha \underline{v} \times \underline{\omega}_e \cdot \frac{\partial}{\partial \underline{v}} + \frac{1}{m_i} \underline{M}(\underline{x}) \cdot \frac{\partial}{\partial \underline{v}} \right) f_i(\underline{\Omega}) \quad (39)$$

III. PROPOSED METHOD OF SOLUTION

Equations (22)-(25) and (36), (37) have the properties of being linear in the φ 's but nonlinear in the f 's. We have in mind, at present, two general schemes for solving these equations.

(A) Since the equations are linear in the correlations, we could solve for them in terms of f_e and f_i by the use of transform methods. This would leave

a set of two coupled non-linear integral equations from which f_e and f_i would be determined. In order to simplify matters in the first calculation we limit consideration to a plane geometry. Let us consider an infinite plane situated at $x = 0$, and let the external medium be the half-space $x > 0$. The half-space $x < 0$ corresponds to the interior of the vehicle. This model would be good at distances small compared to the curvature of the vehicle surface, and is therefore best for large bodies. Thus, these calculations will give results which represent the opposite extreme to those usually reported in the literature, which are valid for bodies small compared to the Debye length. For reviews of this work we refer to Chopra [12] and Zachary [13]. Generalizations to more complicated geometries will be considered after some experience in solving the equations has been obtained. The most interesting situation occurs when the magnetic field is parallel to the plane, i.e., the magnetic field vector lies in the yz -plane. In this case the singlet distributions f_e and f_i do not depend on y or z , which will simplify the equations. To further simplify the situation we shall neglect the time dependence of the quantities of interest. These terms would be of importance in the consideration of oscillatory phenomena in the sheath (plasma oscillations, etc.) and also in the consideration of the effect of radio frequency radiation on the sheath properties. We defer these considerations until a later date.

Using the simplifications discussed above, equations (22)-(25) and (36), (37) can be written in a somewhat simpler form than was done earlier. However, the major simplification is, at this stage of the analysis, somewhat implicit. That is, since the only space coordinate involved is x , the equations found for f_e and f_i , by elimination of the ϕ 's, will be one-dimensional. This

will permit a simpler treatment than would otherwise be possible. Therefore, since a compact vector notation was used, the simplifications discussed above will not be evident at the present stage of the analysis. For this reason it is not of much value to write down the new equations.

There are many types of integral transforms which could be used to solve for the ϕ 's. We choose to use Fourier transforms on the velocity space, and either Hankel or Mellin transforms on the x -coordinate.

Once the nonlinear equations for f_e and f_i are obtained, the plan is to solve them by the use of a variational method. This method will involve the use of a trial function for the f 's. We actually do not require knowledge of f_e and f_i , but only certain integrals involving these quantities will be needed. For example, the particle density distributions are given by integrals of the f 's over velocity space. The scalar potential is given by,

$$\phi(x) = e \iint \frac{[f_i(x', v) - f_e(x', v)]}{|x - x'|} dx' dv \quad (40)$$

subject to the boundary condition that ϕ is constant at $x = 0$ and vanishes as $x \rightarrow \infty$. $\phi(x)$ varies through the sheath, not over the vehicle surface. We would hope to compare $\phi(0)$ with experimental measurements of the vehicle potential. Since the f 's will be integrated over the velocity space to give the quantities of interest, it will be a good approximation to take a modified Gaussian for the velocity dependence, which will also allow the velocity integrations to be easily performed. In addition, the trial functions for the f 's will contain functions of x which must be determined by use of the variational procedure. Thus, we will then have a simpler situation of having a set of one-dimensional equations, even though they will in general still

be nonlinear. However, this does not constitute a serious problem in the variational methods.

The singlet distributions, f_e and f_i , must satisfy further restrictions. Firstly, since we will deal with a time-independent situation, the net current in the sheath must be conserved. This requires that,

$$\nabla \cdot \underline{j} = 0 \quad (41)$$

where:

$$\underline{j} = e \iiint \underline{v} [f_i(x, \underline{v}) - f_e(x, \underline{v})] d\underline{v} \quad (42)$$

The condition (41) will be included in the variational procedure by the use of Lagrange multipliers.

One final point is that the singlet distributions must satisfy certain boundary conditions. Clearly f_e and f_i must remain finite as $x \rightarrow \infty$. Also, some conditions must be imposed at $x = 0$. For example, we may take

$$f(x=0) = f_i(x=0) = 0 \quad (43)$$

(B) Instead of solving for the correlation functions (φ 's) first, and then eliminating them from the remainder of the equations it may turn out to be better to apply the variational method to the equations right at the start. The advantage of this procedure would be that the velocity dependence of the equations is eliminated at the beginning. It is too soon to decide between the methods.

In conclusion, we may say that a calculational scheme which shows signs of being successful is emerging, at least for a simple geometry. This should pave the way towards the treatment of more complicated situations.

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